Weight (uncertainty) propagation in networks of hypernodes

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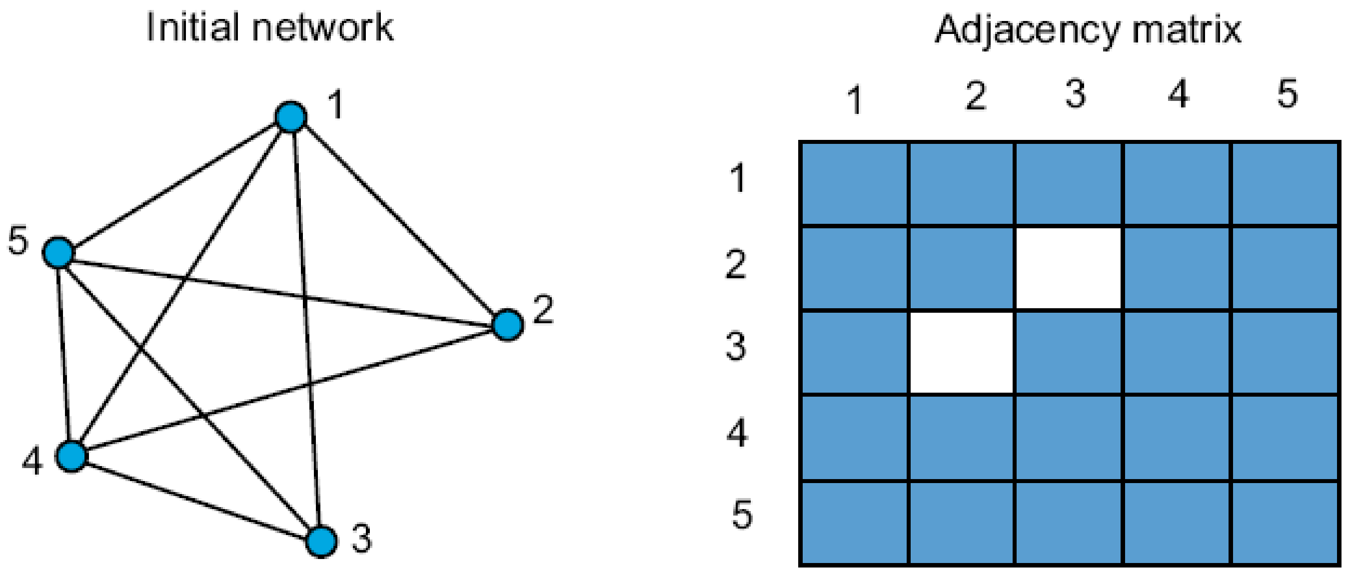
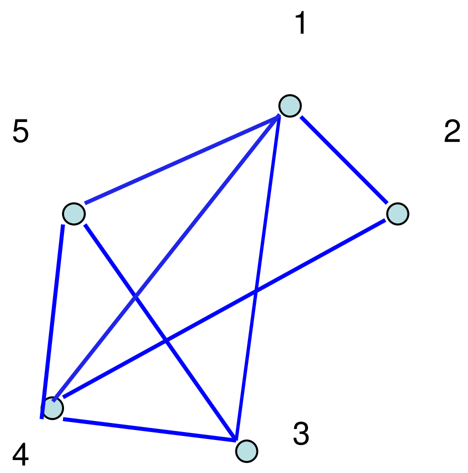
Network representation and its adjacency matrix



Unobserved link between nodes 2 and 3 is represented in white with the value 0

Representation of network with prohibited/unlikely or unobserved Links



Network with no link between
Nodes 2 and 3 and Nodes 2 and 5.

Assume that:

x =1 means there is a link

x= 0 means the link is unlikely/prohibited

y = 1 means a link is observed

y = 0 means a link is not observed

If we assign probability measures as follows:

p(x=1) = 0.9

p(x=0) = 0.1

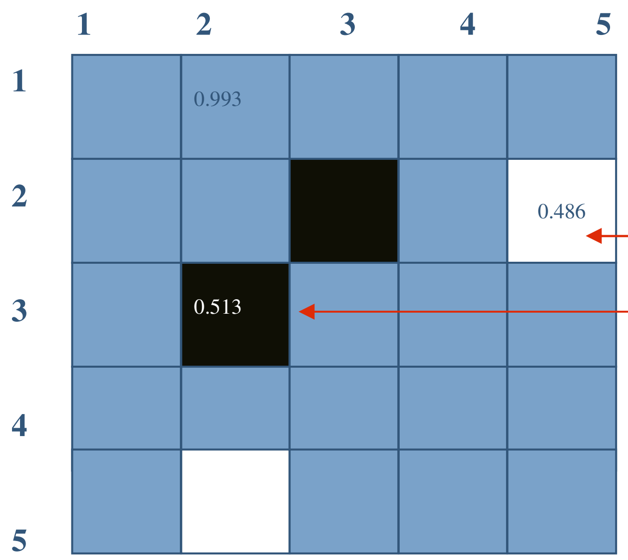
p(y=1|x=1)=0.9

p(y=1|x=0)=0.05

p(y=0|x=1)=0.1

p(y=0|x=0)=0.95

Bayes' Theorem



$p(x=1|y=1) = 0.993$

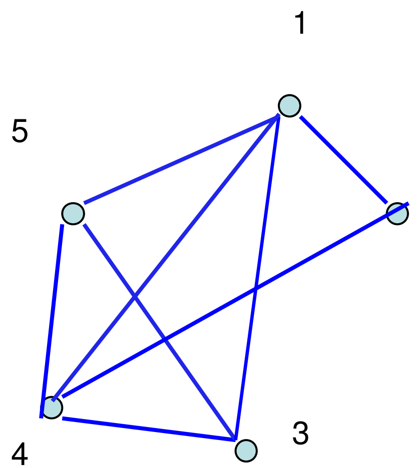
$p(x=1|y=0) = 0.486$

$p(x=0|y=0) = 0.513$

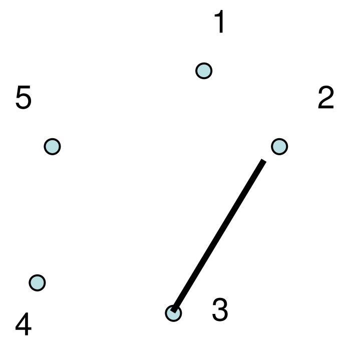
$p(x=0|y=1)=0.006$

Unobserved link between nodes 2 and 5 is represented in white with the value 0.486 but the unobserved link between 2 and 3 is represented in black with the value of ~0.0

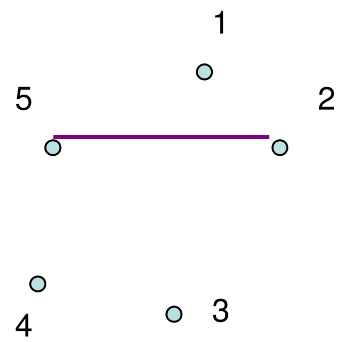
Prohibited/Unlikely and Potential Missing Links



An observed network

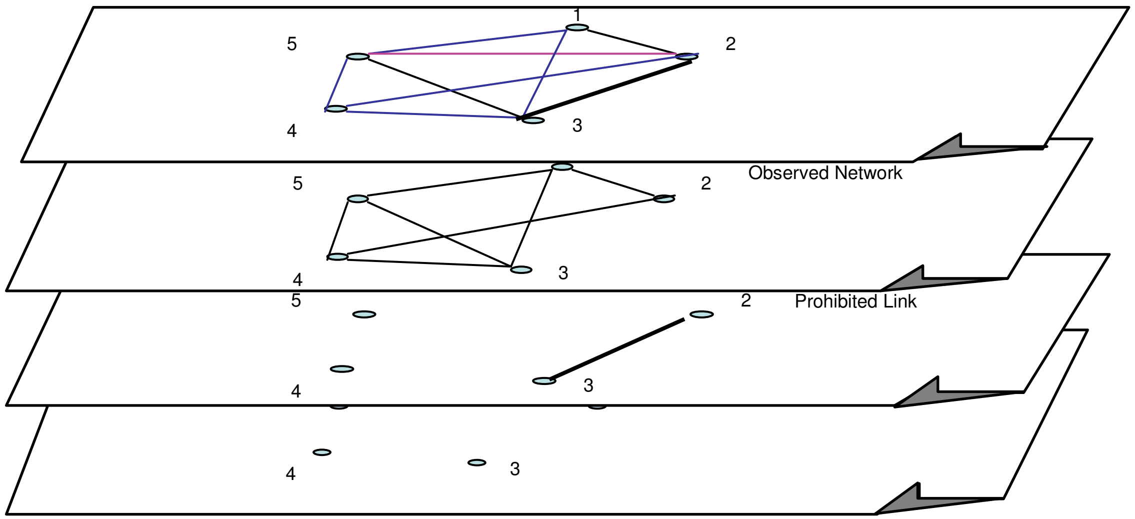


A network with a prohibited/unlikely Link



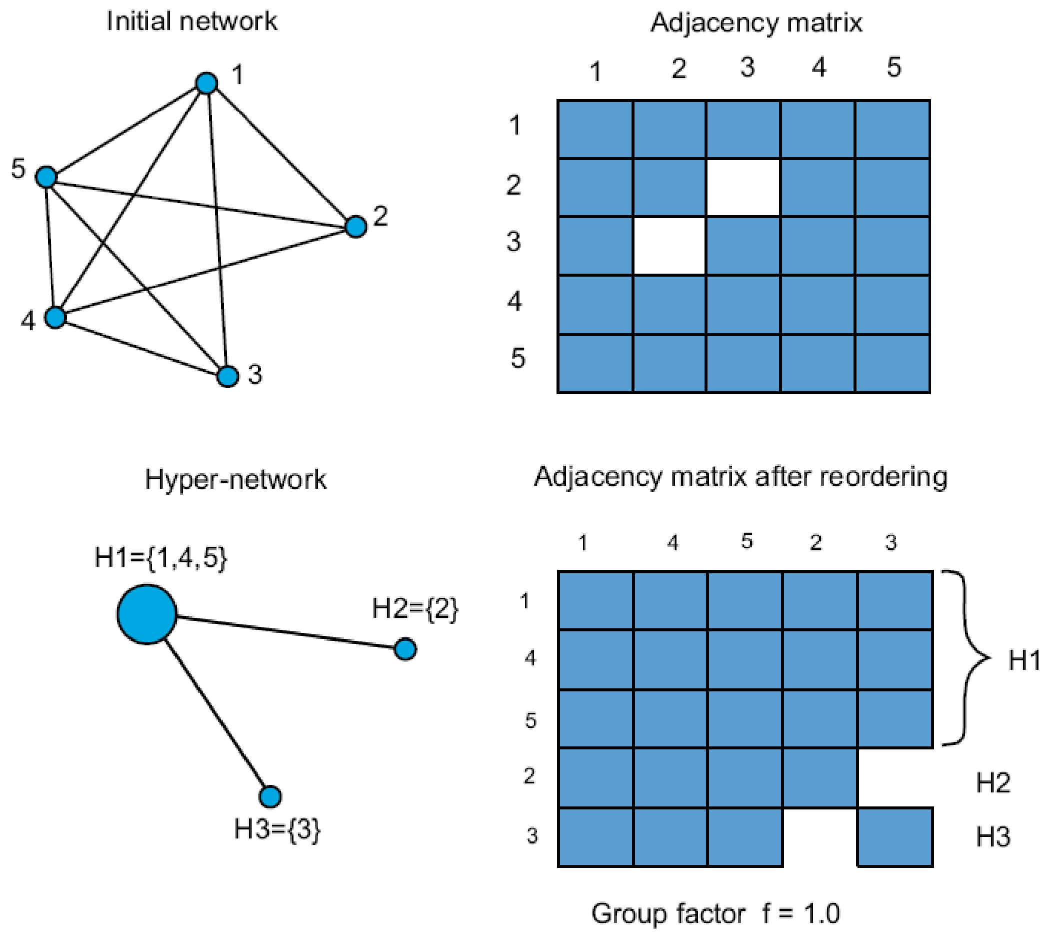
A network with a potential missing Link

Complete Network



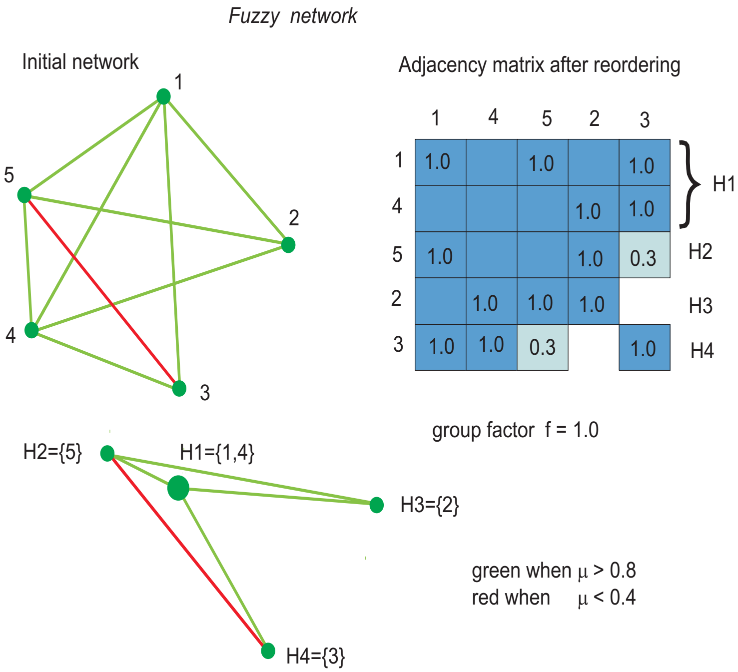
A layered network can be visualised by stacking the 3 networks together to give additional information regarding the underlying structure and pattern of the whole network.

The concept of hypernode from a simple network



Definition 1 A graph

We suppose a graph denoted by G(V,E) nodes (vertices) and |E| edges is a set of nodes and E a set of edges between the nodes



Definition 2 Weighted graph

The graph may be weighted (to represent uncertainties), in which case. We associate a real valued function with an Edge or a node. The weight of a node is modelled as the self-connecting edge *w(i,i)*.

Definition 3 *Hypernodes define a partition*

A collection P of nonemty subsets $P_1, P_2, \cdots P_n$ of V define a partition of V when $P_1 \cup P_2 \cup \cdots \cup P_n = V$ and for any elements of P the intersection $P_i \cap P_j = \emptyset$ for $i \neq j$.

Definition 4 *Edges between hypernodes*

The set of edges between to subsets P_i and P_j of a partition of P is denoted as $E_{i,j}$ and defined from a set of node pairs (u, v) as:

$$E_{i,j} = \cup(u, v), \ \ \forall\ u \in P_i \ \text{ and } \ \forall\ v \in P_j.$$

Definition 5 *Hypernodes, a certain hypernode represents a collapse from a set of nodes into one single node.*

The hypernodes $H = \{h_1, h_2, \cdots, h_n\}$ of a partition $P = \{P_1, P_2, \cdots, P_n\}$ is defined on the basis of a mapping $P \longrightarrow H$ as $\mu(P_i | P_i \in P) = h_i$, where $h_i \in H$.

Definition 6 *Hierarchy of networks*

A hierarchy $\mathbf{T}$ of graphs can be derived as $\mathbf{T} = \{G^0, G^1, \cdots, G^k\}$ where $|G^i| > |G^{i+1}|$ and $|G^k| \geq 1$.

Here, G^0 denotes the original graph and G^k the top level, i.e, the most generalized version of the graph.

The weight of the edges between the hypernodes, the arithmetic mean value

Assume two sets P_i and P_j of a partition P and their corresponding hypernodes h_i and h_j. The arithmetic mean value $w_1(\cdot)$ is computed as

$$w_1(h_i, h_j) = \frac{1}{|E_{i,j}|} \sum_{e \in E_{i,j}} w(e).$$

The minimum or maximum value

$$w_{-\infty}(h_i, h_j) = \min(w(e) \mid e \in E_{i,j})$$

or

$$w_{+\infty}(h_i, h_j) = \max(w(e) \mid e \in E_{i,j}).$$

A Gaussian view

The variance of the average value m of some uncorrelated observations $X = \{x_1, x_2, \cdots, x_n\}$ can be computed according to the Gaussian low of error propagation as: $\sigma_m^2 = \Sigma_{i=1}^n \sigma_i^2 / n^2$. The strength q, i.e., the weight of an observation, is computed as the inverse of its variance. This result introduced gives $1/q_m = (\Sigma_{i=1}^n 1/q_i)/n^2$.

By reorganizing we get

$$q_m = n \frac{n}{\Sigma_{i=1}^n 1/q_i} = \beta w_{-1}(Q),$$

where $\beta = n$, $Q = \{q_1, q_2, \cdots, q_n\}$ and $w_{-1}(Q)$ the harmonic mean of the weights.

A Gaussian model

A Gaussian weight $w_g(\cdot)$ of the edges between the hypernodes can therefore be defined as

$$w_g(h_i, h_j) = \beta w_{-1}(e|e \in E_{i,j}),$$
where $\beta = |E_{i,j}|$.

Generalized mean

A class of average operators is defined by the generalized mean $w_\alpha(\cdot)$ as

$$w(h_i, h_j) = \beta w_\alpha(h_i, h_j) = \beta \, [\frac{1}{|E_{i,j}|} \, \Sigma_{e \in E_{i,j}} w(e)^\alpha]^{1/\alpha}, \quad w(e) > 0$$

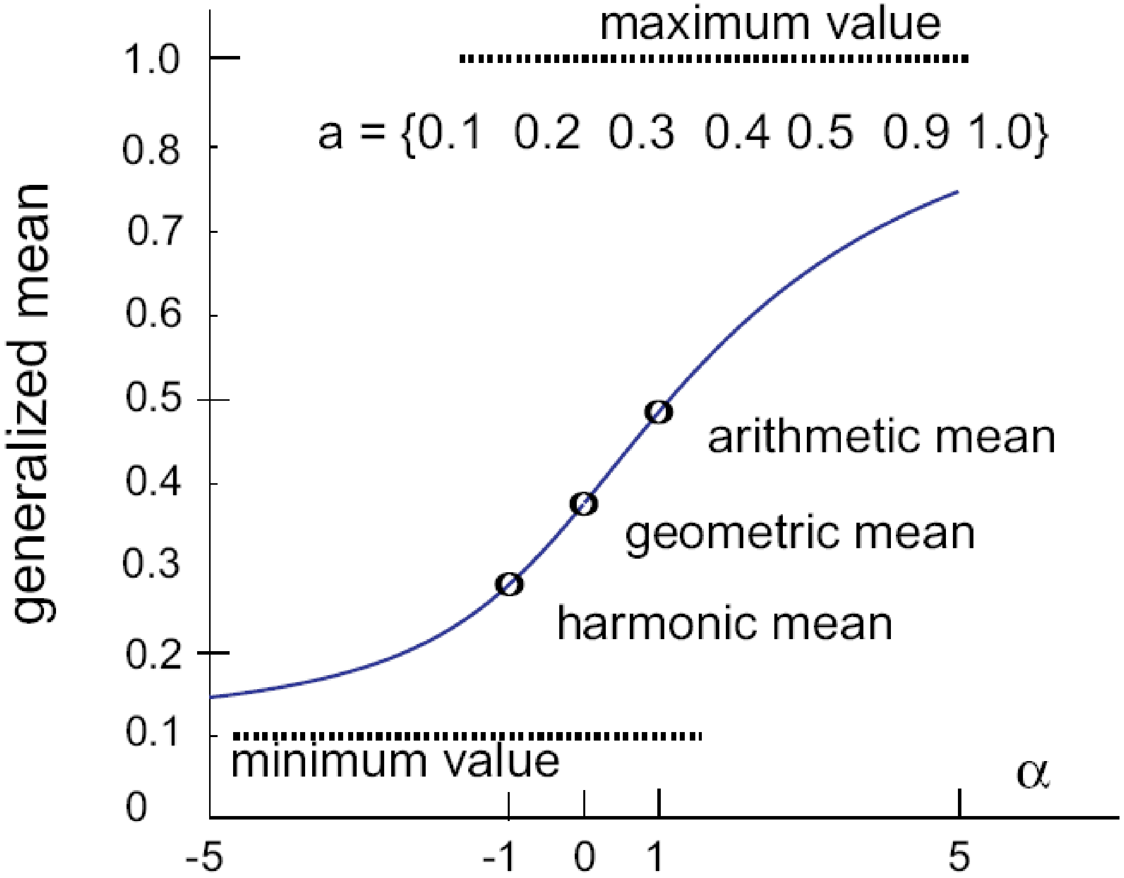
where $\alpha \in \mathcal{R}$ and β is a modifier defined as

β	Interpretation
1	
$\lvert E_{i,j} \rvert$	Gaussian weight when $\alpha = -1$

Generalized mean

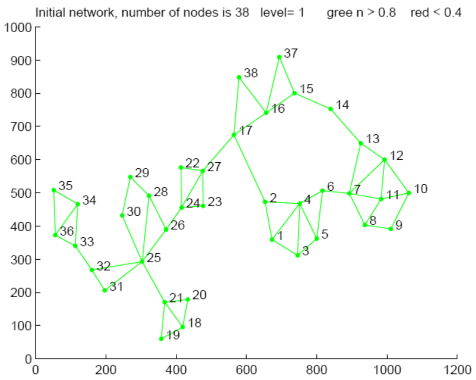
When $\alpha \to -\infty$, $w_\alpha(\cdot)$ corresponds to the minimum operator, $\alpha = -1$ defines the harmonic mean, $\alpha \to 0$ in the limit case approaches the geometric mean, $\alpha = 1$ the arithmetic mean and $\alpha \to \infty$ the maximum operator.

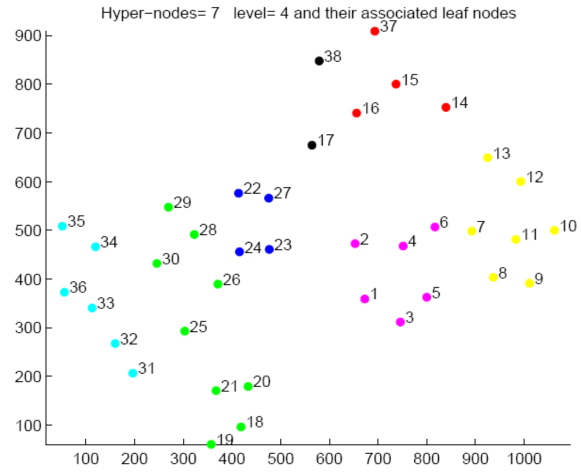
Example on how the generalized mean relates to parameter α

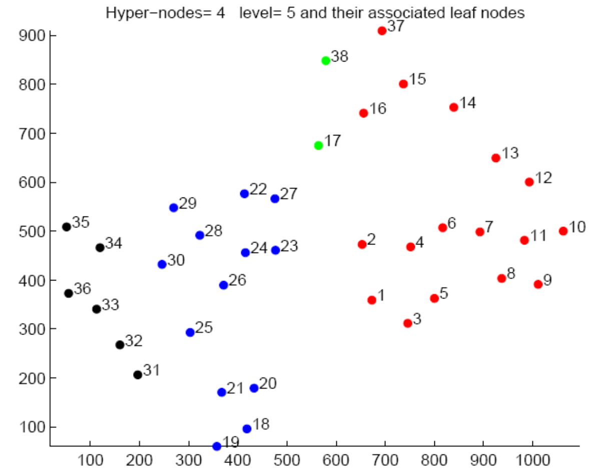


Maximum and Gaussian operator

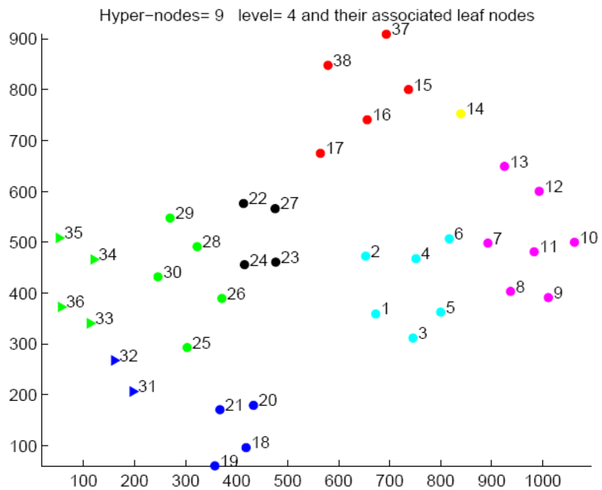
Maximum

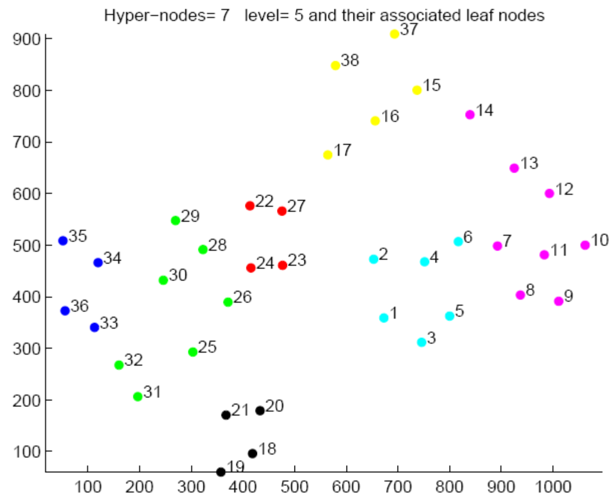




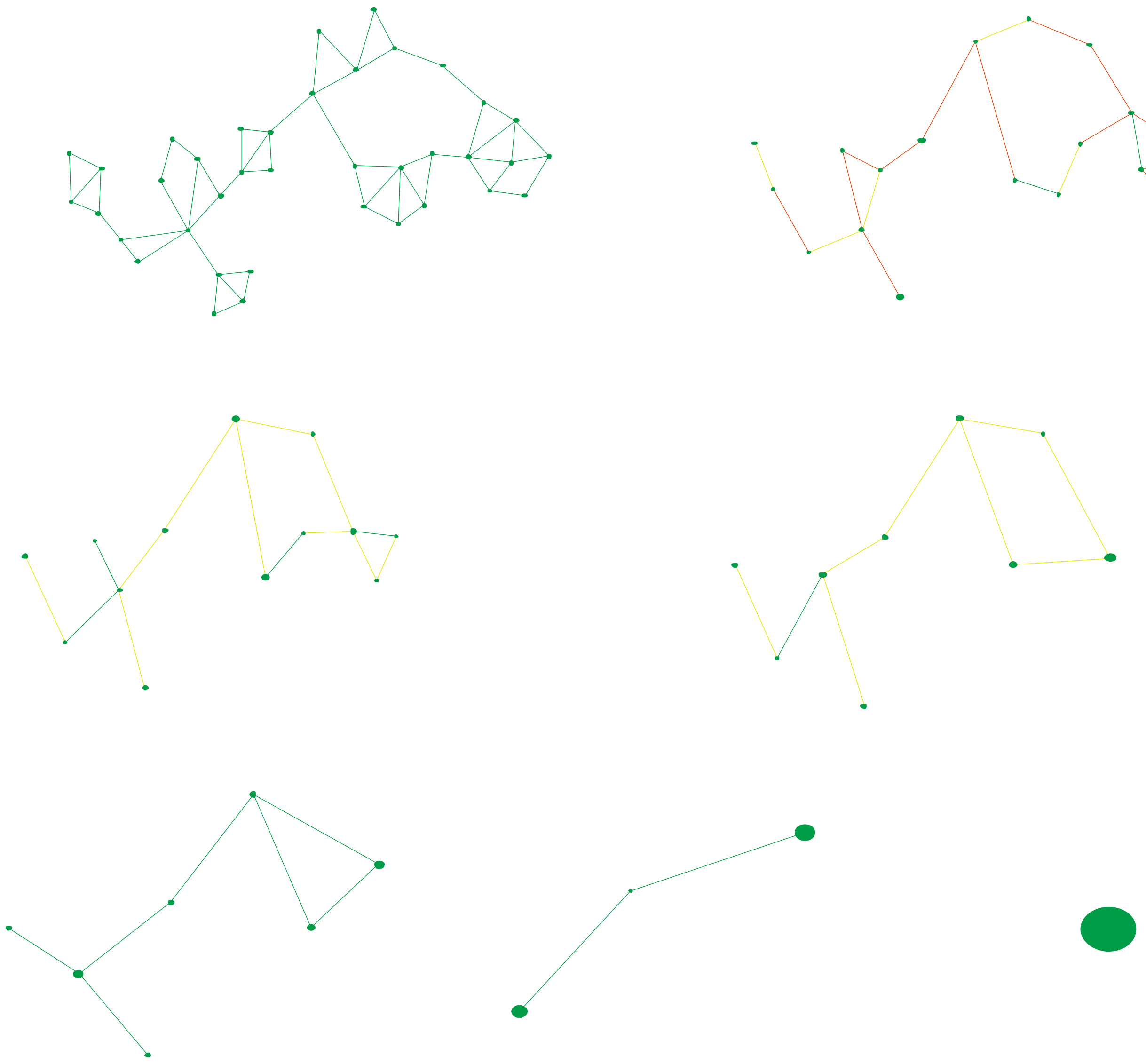


Gaussian





Hierarchy of hypernodes, Gaussian weight propagation model



Other Generalized Linear Models

Model	Link Equation
Poisson	$\log(\lambda_k) = \sum_{j=1}^{J} \theta_j z(x_{k-j})$
Binomial	$\log(\frac{p_k}{1-p_k}) = \sum_{j=1}^{J} \theta_j x_{k-j}$

Conclusions

- A class of operators to compute the weight of the edges between hypernodes is defined on the basis of the generalized mean.
- Selection of the parameters of these operators is an application dependent task.
- Propagation of uncertainties is an important issue

Acknowledgements

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